Transverse stability of Kawahara solitons

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The transverse stability of the planar solitons described by the fifth-order Korteweg–de Vries equation (Kawahara solitons) is studied. It is shown that the planar solitons are unstable with respect to bending if the coefficient at the fifth-derivative term is positive and stable if it is negative. This is in agreement with the dynamics of the two-dimensional Kawahara solitons.

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I. INTRODUCTION

We consider the transverse stability of the planar solitons described by the fifth-order Korteweg–de Vries (KdV) equation

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial u^2}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}+\gamma\frac{\partial^5 u}{\partial x^5}=0 , \tag{1}$$

containing the next order dispersive term. Equation (1) arises in dispersive fluid dynamics (e.g., shallow water waves), plasma physics, etc. The fifth-derivative term may lead to significant qualitative effects not only for sufficiently large

$$\gamma\left[\beta L^2\right]^{-1} ,$$

where L is a characteristic length of nonlinear structures [1,2], but also in cases when this parameter is small [3]. A detailed (numerical) investigation of the solitons described by Eq. (1) was first performed by Kawahara [1], and we call them Kawahara solitons.

The transverse stability of Kawahara solitons can be investigated by means of the (1+2)-dimensional generalization of Eq. (1), which can be written as

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial (u^2)}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}+\gamma\frac{\partial^5 u}{\partial x^5}=\frac{\partial \varphi}{\partial y} , \tag{2a}$$

$$\frac{\partial \varphi}{\partial x}=-\tfrac{1}{2}C\frac{\partial u}{\partial y} , \tag{2b}$$

where $C>0$ is the velocity of linear waves in the limit when the dispersion is neglected. At $\gamma=0$, the system (2) turns into the Kadomtsev-Petviashvili (KP) equation [4], and it can be derived in the same way. Kadomtsev and Petviashvili came to the conclusion [4] that the planar KdV solitons are stable for $\beta>0$ and unstable at $\beta<0$. On the other hand, it is known that the KP solitons are planar in the stable case and "lumps" in the unstable. Taking into account that the existence, structure, and dynamics of the planar Kawahara solitons significantly depend on the signs of γ and β [1–3], one can expect that their stability must also depend on these signs.

It is also interesting to compare the stability conditions of the planar solitons with the properties of solitons and other pulselike solutions to Eqs. (2), i.e., in (1+2) dimensions [5,6], which also depend on the signs of γ and β.

The stability conditions of the planar Kawahara solitons are derived in Sec. II. In Sec. III, they are discussed in connection with the properties of two-dimensional solitons and pulselike solutions to Eqs. (2).

II. STABILITY CONDITIONS

We look for a solution of Eqs. (2) of the form

$$u=a[f_0(\xi)+\lambda^2 f_2(\xi,Y,T)] , \tag{3}$$

where

$$\xi=|a/\beta|^{1/2}(x-x_0) , \quad dx_0/dt=a(Y,T) , \tag{4a}$$

$$Y=\lambda y , \quad T=\lambda^2 t , \tag{4b}$$

λ is a small parameter, and $f_0(\xi)$ satisfies the boundary conditions

$$\xi^n f_0(\xi)\to 0 \quad (|\xi|\to\infty , \quad n>0) . \tag{5}$$

The dimensionless function $f_0(\xi)$ describes, as will be confirmed below, the shape of a planar soliton. The soliton bending is characterized by the dependence of a on stretched variables Y,T and the terms $O(\lambda^2)$ in (3).

Substituting (3) into (2b), we have

$$\frac{\partial\varphi}{\partial\xi}=\tfrac{1}{2}\lambda C\left[a\frac{\partial f_0}{\partial\xi}\frac{\partial x_0}{\partial Y}-\left|\frac{\beta}{a}\right|^{1/2}\left[f_0+\tfrac{1}{2}\xi\frac{\partial f_0}{\partial\xi}\right]\frac{\partial a}{\partial Y}\right] , \tag{6}$$

where we have used

$$\left\{\frac{\partial[af_0(\xi)]}{\partial a}\right\}_{x,x_0}=f_0(\xi)+\tfrac{1}{2}\xi\frac{\partial f_0}{\partial\xi} . \tag{7}$$

Integration of (6) gives

$$\varphi=\tfrac{1}{2}\lambda C\left\{af_0(\xi)\frac{\partial x_0}{\partial Y}-\frac{1}{2}\left|\frac{\beta}{a}\right|^{1/2}\left[\int_{-\infty}^{\xi}d\xi' f_0(\xi')+\xi f_0(\xi)\right]\frac{\partial a}{\partial Y}\right\}+\psi(t,Y,T) , \tag{8}$$

where $\psi(t,Y,T)$ is determined by the condition

$$\varphi(\xi,t,Y,T)=0$$

in front of the soliton, which means that the soliton moves into an unperturbed region. Therefore

$$\phi(\infty,t,Y,T)=0 \quad (a>0)\ , \qquad \phi(-\infty,t,Y,T)=0 \quad (a<0)\ . \tag{9}$$

From (8) and (9) it follows that Ψ does not depend on t and

$$\Psi(Y,T)=\tfrac{1}{8}\lambda C\left|\frac{\beta}{a}\right|^{1/2}(1+\mathrm{sgn}a)\frac{\partial a}{\partial y}\int_{-\infty}^{\infty}d\xi\, f_0(\xi)\ . \tag{10}$$

Differentiating (8) with respect to y and neglecting nonlinear terms, we have

$$\frac{\partial\varphi}{\partial y}=\tfrac{1}{2}\lambda^2 C\left\{af_0(\xi)\frac{\partial^2 x_0}{\partial Y^2}-\frac{1}{2}\left|\frac{\beta}{a}\right|^{1/2}\times\left[\int_{-\infty}^{\xi}d\xi' f_0(\xi')+\xi f_0(\xi)-\tfrac{1}{2}(1+\mathrm{sgn}a)\int_{-\infty}^{\infty}d\xi\, f_0(\xi)\right]\frac{\partial^2 a}{\partial Y^2}\right\}\ . \tag{11}$$

Substituting (3) and (11) into (2a) and collecting terms $O(1)$, we have

$$\mathrm{sgn}(\beta\gamma)\varepsilon^2\frac{d^4 f_0}{\partial\xi^4}+\frac{d^2 f_0}{d\xi^2}+\mathrm{sgn}(\beta a)\left[\tfrac{1}{2}f_0^2-f_0\right]=0\ , \tag{12a}$$

$$\varepsilon^2=|\gamma a|\beta^{-2}\ . \tag{12b}$$

Equations (12) with boundary conditions (5) define, as it was assumed above, soliton solutions of the fifth-order KdV equation (1), i.e., the Kawahara solitons, which exists at [1,3]

$$\gamma\beta>0\ ,\quad a\beta<0\ ,\quad \varepsilon>\tfrac{1}{2}\ , \tag{13a}$$

$$\gamma\beta<0\ ,\quad a\beta>0\ ,\quad 0<\varepsilon<\infty\ . \tag{13b}$$

In both cases (13) with $\varepsilon>\frac{1}{2}$, the solitons have oscillatory structures [1] and in the case

$$\gamma\beta>0\ ,\quad a\beta>0\ ,\quad \varepsilon\ll 1\ , \tag{14}$$

the solitons are quasistationary, radiating waves with the phase velocity equal to the soliton velocity [3]. One can also show that $f_0(\xi)=f_0(-\xi)$.

Collecting terms $O(\lambda^2)$ in (2a), and using (7), we have

$$\left[f_0+\tfrac{1}{2}\xi\frac{\partial f_0}{\partial\xi}\right]\frac{\partial a}{\partial T}+\left|\frac{a}{\beta}\right|^{3/2}\beta a\left[\mathrm{sgn}(\beta a)\frac{\partial}{\partial\xi}(f_0f_2-f_2)+\frac{\partial^3 f_2}{\partial\xi^3}+\mathrm{sgn}(\gamma\beta)\varepsilon^2\frac{\partial^5 f_2}{\partial\xi^5}\right]$$
$$=\tfrac{1}{2}C\left\{af_0\frac{\partial^2 x_0}{\partial Y^2}-\frac{1}{2}\left|\frac{\beta}{a}\right|^{1/2}\left[\int_{-\infty}^{\xi}d\xi' f_0(\xi')+\xi f_0(\xi)-\tfrac{1}{2}(1+\mathrm{sgn}a)\int_{-\infty}^{\infty}d\xi\, f_0(\xi)\right]\frac{\partial^2 a}{\partial Y^2}\right\}\ . \tag{15}$$

Multiplying both sides of (15) by $f_0(\xi)$, then integrating from $\xi=-\infty$ to $\xi=\infty$, and taking into account (12) and

$$\int_{-\infty}^{\infty}d\xi\, f_0(\xi)\left[f_0(\xi)+\tfrac{1}{2}\xi\frac{\partial f_0(\xi)}{\partial\xi}\right]=\tfrac{3}{4}\int_{-\infty}^{\infty}d\xi\, f_0^2(\xi)\ , \tag{16}$$

$$\int_{-\infty}^{\infty}d\xi\, f_0(\xi)\left[\int_{-\infty}^{\xi}d\xi' f_0(\xi')+\xi f_0(\xi)\right]=\tfrac{1}{2}\left[\int_{-\infty}^{\infty}d\xi\, f_0(\xi)\right]^2\ , \tag{17}$$

we have

$$\left[\frac{\partial a}{\partial T}-\frac{2aC}{3}\frac{\partial^2 x_0}{\partial Y^2}\right]\int_{-\infty}^{\infty}d\xi\, f_0^2(\xi)+\tfrac{1}{6}C\frac{\partial^2 a}{\partial Y^2}\left|\frac{\beta}{a}\right|^{1/2}(1-\mathrm{sgn}a)\left[\int_{-\infty}^{\infty}f_0(\xi)\,d\xi\right]^2=0\ . \tag{18}$$

Introducing the notation

$$\alpha=\frac{C}{6}\left|\frac{\beta}{a}\right|^{1/2}\left[\int_{-\infty}^{\infty}d\xi\, f_0(\xi)\right]^2\times\left[\int_{-\infty}^{\infty}d\xi\, f_0^2(\xi)\right]^{-1}(1-\mathrm{sgn}a) \tag{19}$$

and taking into account the second of Eqs. (4a), we come to the equation describing the soliton bending in linear approximation:

$$\frac{\partial^2 x_0}{\partial t^2}-\frac{2aC}{3}\frac{\partial^2 x_0}{\partial y^2}+\alpha\frac{\partial^3 x_0}{\partial t\,\partial y^2}=0\ , \tag{20}$$

where a and α are considered as constants. Looking for the solution of Eq. (20) of the form

$$x_0=\mathrm{const}\times\exp(iky-i\omega t)\ , \tag{21}$$

we have the dispersion relation

$$\omega^2-i\alpha k^2\omega-\tfrac{2}{3}aCk^2=0\ , \tag{22}$$

which gives

$$\omega=\tfrac{1}{2}k^2[i\alpha\pm(\tfrac{8}{3}aC-\alpha^2k^2)^{1/2}]\ , \tag{23}$$

where, according to (19), $\alpha=0$ for $a>0$ and $\alpha>0$ for $a_0<0$. Therefore, solitons are stable at $a>0$ and unstable at $a<0$. Taking into account (13), we conclude that the planar Kawahara solitons are unstable with respect to bending if

$$\gamma>0\ ,\quad \beta>0\ ,\quad \varepsilon>\tfrac{1}{2}\ , \tag{24a}$$

and

$$\gamma>0\ ,\quad \beta<0\ ,\quad 0<\varepsilon<\infty\ . \tag{24b}$$

As far as one of the conditions (13) must be satisfied, one can write, instead of (24), the instability condition as

$$\gamma>0\ . \tag{25}$$

In the case

$$\gamma<0\ , \tag{26}$$

the planar Kawahara solitons are stable in our approximation, i.e., taking into account terms $O(\lambda^2)$.

III. DISCUSSION

Our results, represented by (24)–(26), are in accord with the properties of stationary solitons and nonstationary pulses in (1+2) dimensions following from Eqs. (2). When the conditions (24) are held, i.e., the planar Kawahara solitons are unstable, Eqs. (2) have stable stationary (1+2)-dimensional soliton solutions [6] decaying in both directions x and y at

$$r=(x^2+y^2)^{1/2}\to\infty\ .$$

(Their stability follows from numerical experiments [6] where it has been shown how they arise from the initial pulselike disturbances and how the larger soliton passes through the smaller one in a collision process.) In case (24a), the (1+2)-dimensional solitons have, like planar Kawahara solitons [1], oscillatory structures in the direction of propagation (parallel to the x axis), and their cross sections, parallel to the y axis, are humps with algebraic asymptotic behavior at $|y|\to\infty$. For $\varepsilon\gg1$, the oscillation periods are small and, after averaging, one comes also to algebraic asymptotic behavior at $|x|\to\infty$ [6]. In case (24b), and $\varepsilon<\frac{1}{2}$, the (1+2)-dimensional solitons have the humplike cross sections both in x and y planes with algebraic asymptotic behavior [6]. (The planar Kawahara solitons are also humps in this case [1].)

In case (26), when the planar Kawahara solitons are stable, there are no stationary (1+2)-dimensional solitons, decaying both in x and y directions. Instead, the pulselike solutions of Eqs. (2) are wave packets, spreading with time. In the x direction they have oscillatory structures and in the y direction they are humps [6].

While comparing the above conclusions (the planar Kawahara solitons are stable at $a>0$ and unstable at $a<0$) with the (1+2)-dimensional solutions of Eqs. (2) described in Ref. [6], it is important to take into account that at $\gamma\beta>0$, $a\beta>0$ there are no stationary planar solitons [3]. Then one has a complete conciliation of the stability conditions, derived above, with the (1+2)-dimensional solutions [6] in a sense that the situation is the same as for the KdV-KP equations: When the planar KdV solitons are stable ($\beta>0$), they are the only soliton solutions of the KP equations; if the planar KdV solitons are unstable ($\beta<0$), the KP solitons are lumps with algebraic asymptotics at $r\to\infty$.

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